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Solution of the Zero Rest-Mass Equations for Free Fields in the Einstein Static Universe

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Abstract

Solutions to the source-free spin s zero rest-mass equations in the Einstein static universe are obtained by means of the Newman–Penrose spin coefficient formalism.

1. *Introduction*

One of the interesting features of the source-free spin s zero rest-mass equations (Penrose, 1965)

$$\nabla^{AX'}\Phi_{AB\ldots K} = 0 \qquad (s \neq 0) \tag{1.1a}$$

$$\Box\,\Phi \equiv (\nabla^{AX'}\nabla_{AX'} + \tfrac{1}{6}R)\,\Phi = 0 \qquad (s = 0) \tag{1.1b}$$

is their invariance under conformal rescalings of the space-time metric

$$ds^2 \to d\hat{s}^2 = \Omega^{-2} ds^2 \tag{1.2}$$

if one assigns to the $\Phi_{AB\ldots K}$ the transformation law

$$\Phi_{AB\ldots K} \to \hat{\Phi}_{AB\ldots K} = \Omega^{s+1}\Phi_{AB\ldots K} \tag{1.3}$$

The spinor fields $\Phi_{AB\ldots K}$ of (1.1) and (1.3) are totally symmetric in the $2s$ spinor indices $A, B, \ldots, K$. From the point of view of (1.1), all background space-times which differ only by a scale transformation (1.2) are equivalent; solutions in any fixed space-time induce corresponding solutions in all other conformally related space-times via (1.3). For most problems in conformally flat background space-times, equations (1.1) are most easily solved in Minkowski space. But when the background space-time is not actually flat, or when given boundary conditions can be simplified by a scale transformation, it may be more convenient to work in other conformally flat space-times.

In this paper equations (1.1) are examined for the Einstein static universe $\mathcal{E}$, with the (conformally flat) metric

$$\begin{aligned} ds^2_{\mathcal{E}} &= d\chi^2 - d\psi^2 - \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2) \\ ds^2_{\mathcal{E}} &= d\chi^2 - \tfrac{1}{4}(d\alpha^2 + d\beta^2 + d\gamma^2 + 2\cos\beta d\alpha d\gamma) \end{aligned} \tag{1.4}$$

where

$$\cos\psi - i\sin\psi\cos\theta = \cos\frac{\beta}{2}\exp\left\{-i\left(\frac{\alpha+\gamma}{2}\right)\right\} \qquad \phi = \frac{\alpha-\gamma}{2}$$

When equations (1.1) are formulated in terms of the infinitesimal operators for the symmetry group of $\mathcal{E}$, they break up into coupled pairs, from which decoupled second-order equations may be derived. The second-order equations are separable in the $\chi, \alpha, \beta, \gamma$ coordinate system, and all separated solutions are easily constructed. By making repeated use of the commutation relations of the infinitesimal operators, it is possible to satisfy the coupled first-order equations as well. This procedure will be carried out in detail below.

2. *The Zero Rest-Mass Equations in* $\mathcal{E}$

The metric (1.4) admits seven independent Killing vector fields,

$$\begin{aligned}
\tau^\mu &\equiv \tfrac{1}{2}\,(1, 0, 0, 0) \\
\xi_1^\mu &\equiv (0, \cot\beta\sin\alpha, -\cos\alpha, -\csc\beta\sin\alpha) \\
\xi_2^\mu &\equiv (0, -\cot\beta\cos\alpha, -\sin\alpha, \csc\beta\cos\alpha) \\
\xi_3^\mu &\equiv (0, -1, 0, 0) \\
\eta_1^\mu &\equiv (0, \csc\beta\sin\gamma, \cos\gamma, -\cot\beta\sin\gamma) \\
\eta_2^\mu &\equiv (0, -\csc\beta\cos\gamma, \sin\gamma, \cot\beta\cos\gamma) \\
\eta_3^\mu &\equiv (0, 0, 0, -1).
\end{aligned}$$

One of these, τ^μ, is time-like and orthogonal to the χ = constant hypersurfaces, while the remaining six, ξ_a^μ and η_a^μ $(a = 1, 2, 3)$, lie entirely within the χ = constant hypersurfaces. The associated differential operators,

$$T \equiv i\tau^\mu \frac{\partial}{\partial x^\mu}$$

$$L_a \equiv i\xi_a^\mu \frac{\partial}{\partial x^\mu}$$

$$M_a \equiv i\eta_a^\mu \frac{\partial}{\partial x^\mu}$$

satisfy

$$\begin{aligned}[L_a, L_b] &= i\epsilon_{abc}L_c \\ [M_a, M_b] &= i\epsilon_{abc}M_c \\ [L_a, M_b] &= 0\end{aligned} \tag{2.1a}$$

and

$$[L_a, T] = [M_a, T] = 0 \tag{2.1b}$$

where ϵ_{abc} is totally antisymmetric with $\epsilon_{123} = 1$. The commutation relations (2.1a) characterise the Lie algebra of $O(4)$, the symmetry group of the space-like 3-spheres χ = constant. With

$$L_\pm \equiv L_1 \pm iL_2$$
$$M_\pm \equiv M_1 \pm iM_2$$

one also has

$$\begin{aligned}[L_3, L_\pm] &= \pm L_\pm \qquad & [M_3, M_\pm] &= \pm M_\pm \\ [L_+, L_-] &= 2L_3 & [M_+, M_-] &= 2M_3\end{aligned} \tag{2.2}$$

In order to make full use of the symmetries of & in dealing with (1.1), it will be convenient to introduce a null tetrad $(l^\mu, n^\mu, m^\mu, \overline{m}^\mu)$ and an associated spinor dyad (o^A, i^A) given by

$$\begin{aligned}l^\mu &\equiv \sqrt{(2)}\,(\tau^\mu + \xi_3^\mu) \longleftrightarrow o^A \bar{o}^{A'} \\ n^\mu &\equiv \sqrt{(2)}\,(\tau^\mu - \xi_3^\mu) \longleftrightarrow i^A \bar{i}^{A'} \\ m^\mu &\equiv \sqrt{(2)}\,(\xi_1^\mu + i\xi_2^\mu) \longleftrightarrow o^A \bar{i}^{A'}\end{aligned}$$

The null Killing vectors l^μ, n^μ, m^μ, and $\overline{m}^\mu$, satisfy the standard orthogonality relations:

$$l^\mu n_\mu = -\, m^\mu \overline{m}_\mu = 1$$

and all other scalar products vanish.

With the above conventions, equations (1.1a) may be written

$$\begin{aligned}\left[T + L_3 + \frac{s+1}{2}\right] \Phi_{(N+1)} &= L_- \Phi_{(N)} \\ \left[T - L_3 + \frac{s+1}{2}\right] \Phi_{(N)} &= L_+ \Phi_{(N+1)}\end{aligned} \qquad (0 \leqslant N \leqslant 2s-1) \tag{2.3}$$

The $\Phi_{(N)}$ in (2.3) are the dyad components of the $\Phi_{AB\ldots K}$,

$$\Phi_{(N)} \equiv \Phi_{AB\ldots RS}\, o^A o^B \ldots i^R i^S$$

where N is the number of times i^A appears on the right-hand side above. From (2.3) one immediately obtains the decoupled second-order equations

$$\left[T^2 + sT - L^2 + \frac{s^2-1}{4}\right] \Phi_{(N)} = 0 \qquad (0 \leqslant N \leqslant 2s) \tag{2.4}$$

where

$$L^2 \equiv L_1^2 + L_2^2 + L_3^2 = L_+L_- + L_3^2 - L_3$$

If one identifies Φ with $\Phi_{(0)}$, equation (1.1b) takes the form of (2.4) with $s = 0$.

3. *Solutions*

Solutions to (2.4) of the form

$$\Phi_{(N)} = S(\chi)A(\alpha)B(\beta)C(\gamma) \tag{3.1}$$

may readily be obtained. One finds, on substituting (3.1),

$$S(\chi) = e^{-2i\tau_\pm\chi}$$

$$A(\alpha) = e^{im\alpha}$$

$$C(\gamma) = e^{im'\gamma}$$

$$B(\beta) = \left(\sin\frac{\beta}{2}\right)^{|m-m'|}\left(\cos\frac{\beta}{2}\right)^{m+m'} W_l\left(\sin^2\frac{\beta}{2}\right)$$

where l, m, m', and

$$\tau_\pm = -\frac{s}{2} \pm (l + \tfrac{1}{2})$$

are the separation constants and W_l must satisfy the hypergeometric equation

$$\omega(1-\omega)\frac{d^2W_l}{d\omega^2} + [1 + |m - m'| - 2\omega(M+1)]\frac{dW_l}{d\omega} + [l(l+1) - M(M+1)]W_l = 0 \tag{3.2}$$

with

$$\omega \equiv \sin^2\frac{\beta}{2}$$

$$M \equiv \frac{|m - m'| + (m + m')}{2} = \begin{cases} m & m \geqslant m' \\ m' & m' \geqslant m \end{cases}$$

Solutions with physically acceptable behaviour in the angular coordinates α, β, and γ occur only for values of m and m' satisfying

$$m - m' = 0, \pm 1, \pm 2, \ldots$$

$$m + m' = 0, \pm 1, \pm 2, \ldots$$

In this case the general solution to (3.2) is (Morse & Feshbach, 1953)

$$W_l(\omega) = AF(M - l, M + l + 1; 1 + |m - m'|; \omega) + BG(M - l, M + l + 1; 1 + |m - m'|; \omega)$$

where $F(a, b; c; \omega)$ is the hypergeometric function and

$$G(a, b; c; \omega) \equiv F(a, b; c; \omega) \int [F(a, b; c; \omega')]^{-2} \omega'^{-c}(1 - \omega')^{c-a-b-1} d\omega'$$

Thus the basic separated solutions, from which more general solutions to (2.4) may be constructed, are

$$\Phi_{\tau_\pm lmm'n} \equiv e^{-2i\tau_\pm\chi} e^{im\alpha} e^{im'\gamma} \left(\sin\frac{\beta}{2}\right)^{|m-m'|} \left(\cos\frac{\beta}{2}\right)^{m+m'} \times \begin{cases} F\left(M - l, M + l + 1; 1 + |m - m'|; \sin^2\frac{\beta}{2}\right) & (n = 1) \\ G\left(M - l, M + l + 1; 1 + |m - m'|; \sin^2\frac{\beta}{2}\right) & (n = 2) \end{cases} \quad (3.3)$$

The $\Phi_{\tau_\pm lmm'n}$ of (3.3) are eigenfunctions of the operators T, L^2, L_3, and M_3, corresponding to the eigenvalues $\tau_\pm$, $l(l + 1)$, m, and m' respectively. $L_\pm$ and $M_\pm$ transform the $\Phi_{\phi_\pm lmm'n}$ among themselves. Consequently the $\Phi_{\tau_\pm lmm'n}$ form a basis for representations of the group of symmetry transformations of & (cf. Kyriakopoulos, 1974).

Unless $s = 0$, the coupled equations (2.3) must still be satisfied. From the commutation relations (2.1)-(2.2), it follows that $L_-^k \Phi_{\tau_\pm lmm'n}$ is an eigenfunction of the operators T, L^2, L_3, and M_3 corresponding to the eigenvalues $\tau_\pm$, $l(l + 1)$, $m - k$, and m' respectively, that is, a linear combination of $\Phi_{\tau\pm lm-km'1}$ and $\Phi_{\tau_\pm lm-km'2}$. Consequently if

$$\Phi_{(0)} = \Phi_{\tau_+ lmm'n} \quad (3.4a)$$

one can satisfy (2.3) by taking

$$\Phi_{(N)} = \frac{L_-^N \Phi_{\tau_+ lmm'n}}{(l + m)(l + m - 1) \ldots (l + m + 1 - N)}, \qquad N = 1, 2, \ldots 2s \quad (3.5a)$$

while if

$$\Phi_{(0)} = \Phi_{\tau_- lmm'n} \quad (3.4b)$$

(2.3) has the solution

$$\Phi_{(N)} = \frac{(-)^N L_-^N \Phi_{\tau_- lmm'n}}{(l - m + 1)(l - m + 2) \ldots (l - m + N)}, \qquad N = 1, 2, \ldots 2s \quad (3.5b)$$

When $\Phi_{(0)}$ is a linear combination of $\Phi_{\tau_\pm lmm'n}$, the $\Phi_{(N)}$ are the corresponding linear combinations of the right-hand sides of (3.5a) and (3.5b).

To the particular solutions (3.5) for $\Phi_{(N)}$, $N = 1, 2, \ldots, 2s$ one may freely add any solution $\Psi_{(N)}$ of the equations

$$\left[T + L_3 + \frac{s + 1}{2}\right] \Psi_{(N)} = 0$$
$$L_+ \Psi_{(N)} = 0 \quad (3.6)$$

at any N level. For equations (3.6) guarantee that, if $\Psi_{(N)}$ is added to any $\Phi_{(N)}$, no change in $\Phi_{(N-1)}, \Phi_{(N-2)}, \ldots, \Phi_{(0)}$ is required to maintain (2.3). By (3.6), $\Psi_{(N)}$ must be a linear combination of $\Phi_{\tau_\pm l_\pm mm'1}$ and $\Phi_{\tau_\pm l_\pm mm'2}$, with

$$l_+(m) \equiv -m-1$$

$$l_-(m) \equiv m$$

One finds, in fact,

$$\Psi_{(N)} = \sum_{mm'} A_{(N)mm'} \Psi_{mm'}$$

where

$$\Psi_{mm'} \equiv e^{i(s+1+2m)\chi} e^{im\alpha} e^{im'\gamma} \left(\sin\frac{\beta}{2}\right)^{m-m'} \left(\cos\frac{\beta}{2}\right)^{m+m'}$$

The coefficients $A_{(N)mm'}$ are arbitrary for each N level. Terms proprotional to $\Psi_{mm'}$ cannot occur in the particular solutions (3.5), for when $l = l_\pm(m-1)$ the denominators on the right vanish, while the numerators containing the $\Psi_{mm'}$ do not. But once such terms are introduced at any N level, they generate terms at higher N levels like the other $\Phi_{\tau_\pm lmm'n}$. If, for instance

$$\Phi_{(N)} = L_-^k \Psi_{(N)}$$

one has from (3.5)

$$\Phi_{(N+1)} = -\frac{L_-^{k+1}\Psi_{(N)}}{k+1}.$$

When

$$\Phi_{(N)} = \Phi_{\tau_\pm lmm'n}$$

the general expression for $\Phi_{(N+1)}$ is

$$\Phi_{(N+1)} = \pm \frac{L_- \Phi_{\tau_\pm lmm'n}}{l - l_\pm(m-1)} + \Psi_{(N+1)}$$

One may summarise the above results as follows. With

$$\Phi_{(0)} = \Phi_{\tau_\pm lmm'n} \tag{3.7a}$$

the general solution for $\Phi_{(N)}$ $(N = 1, 2, \ldots, 2s)$ is

$$\Phi_{(N)} = \frac{(\pm)^N L_-^N \Phi_{\tau_\pm lmm'n}}{\prod_{k=0}^{N-1} [l - l_\pm(m-k-1)]} + \sum_{k=1}^{N} \frac{(-)^{N-k}}{(N-k)!} L_-^{N-k} \Psi_{(k)} \tag{3.7b}$$

From the solution (3.7) with $\Phi_{(0)}$ in separated form, all solutions to equations (1.1) in & may be obtained.

References

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